Quantum Noise in Optical Communication Systems

Haiqing Wei[1,2,*] and David V. Plant[2]

[1]*Synopsys, Inc., Mountain View, CA 94043*
[2]*McGill University, Montreal, Canada H3A-2A7*
[*]*hwei1@po-box.mcgill.ca*

ABSTRACT

It is noted that the fiber propagation loss is a random process along the length of propagation. The stochastic nature of the loss process induces a random fluctuation to the energy of the optical signals, which, as an extra source of noise, could become comparable to the amplified-spontaneous-emission noise of optical amplifiers. The optical noise in random loss/gain has a quantum origin, as a manifestation of the corpuscular nature of electromagnetic radiation. This paper adopts the Schrödinger representation, and uses a density matrix in the basis of photon number states to describe the optical signals and their interaction with the environment of loss/gain media. When the environmental degrees of freedom are traced out, a reduced density matrix is obtained in the diagonal form, which describes the total energy of the optical signal evolving along the propagation distance. Such formulism provides an intuitive interpretation of the quantum-optical noise as the result of a classical Markov process in the space of the photon number states. The formulism would be more convenient for practical engineers, and should be sufficient for fiber-optic systems with direct intensity detection, because the quantity of concern is indeed the number of photons contained in a signal pulse. Even better, the model admits analytical solutions to the photon-number distribution of the optical signals.

Keywords: optical communications, quantum noise, loss and gain, random process, Markov process, photon-number distribution.

1. INTRODUCTORY QUANTUM FIBER OPTICS

In modern fiber transmission lines, the optical signals experience alternating loss and gain. The amplified spontaneous emission (ASE) of the in-line amplifiers is usually blamed and considered as the sole source of noise that corrupts the optical signals. However, it should be noted that the fiber propagation loss is a random process along the length of propagation. The stochastic nature of the loss process induces a random fluctuation to the energy of the optical signals, namely, an extra source of noise, which could become comparable to the commonly blamed ASE noise. It is therefore necessary to understand and include this noise in system design and performance evaluation. Fundamentally, the optical noise in random loss/gain has a quantum origin, incurred as a result of the corpuscular nature of electromagnetic radiation. Such quantum noise is often treated in the Heisenberg representation, and interpreted as the result of a Langevin noise operator,[1] or vacuum field operators,[2,3] added to the Heisenberg field operator of the signal. This paper adopts the Schrödinger representation, and uses a density matrix in the basis of photon number states to describe the signal field, the medium reservoir, and their interactions. When the medium degrees of freedom are traced out, a reduced density matrix is obtained in the diagonal form, which describes the total energy of the optical signal evolving along the propagation distance. Such formulism is sufficient for practical fiber-optic systems with direct intensity detection, because the quantity of concern is indeed the number of photons contained in a signal pulse. Furthermore, our formulism provides a more intuitive interpretation of the quantum-optical noise as the result of a classical Markov process in the space of the photon number states.

To deal with quantum noise, we naturally need the quantum theory of light,[3,4] which was first developed by Dirac in 1927.[5] The established procedure of the so-called canonical quantization of radiation starts from a set of classical modes of the electromagnetic field, then relates each mode to a quantum-mechanical harmonic oscillator, and associates to which two operators, named the annihilation and the creation operators respectively. Both operators have non-negative integers as eigen values. The corresponding eigen states are called the number states, and interpreted as having integer numbers of photons excited in the mode referred. For the problem

Optical Modeling and Performance Predictions, edited by Mark A. Kahan, Proceedings of SPIE
Vol. 5178 (SPIE, Bellingham, WA, 2004) · 0277-786X/04/$15 · doi: 10.1117/12.506472

of signal transmission in an optical fiber, it is convenient to describe the optical signals in terms of the eigen propagation modes of the fiber waveguide. Especially for a single-mode fiber, there is only one guided mode (actually two if counting the different polarizations), optical energy in all other spatial modes are not well-confined in the fiber and eventually get lost into the environment. With a fixed polarization, it is customary to represent an information-carrying optical pulse by,[6, 7]

$$E(z,t) = E_0\text{Re}[A(z,t)\exp(-i\omega_0 t)] = E_0[A_r(z,t)\cos\omega_0 t + A_i(z,t)\sin\omega_0 t], \quad (1)$$

$$H(z,t) = H_0\text{Im}[A(z,t)\exp(-i\omega_0 t)] = H_0[A_i(z,t)\cos\omega_0 t - A_r(z,t)\sin\omega_0 t], \quad (2)$$

where E and H are the electric and magnetic fields respectively, ω_0 is the center frequency of the optical signal, and $A(z,t) = A_r(z,t) + iA_i(z,t)$ is the so-called slow-varying envelope of the signal pulse, that is, $A(z,t)$ has small derivatives with respect to t. The envelope $A(z,t)$ is found to satisfy a nonlinear Schrödinger equation (NLSE),[6, 7]

$$\frac{\partial A}{\partial z} + \beta_1\frac{\partial A}{\partial t} + \frac{i}{2}\beta_2\frac{\partial^2 A}{\partial t^2} = i\beta_0 A + i\gamma|A|^2 A, \quad (3)$$

where β_0 is the propagation constant at the center frequency, β_1 is the inverse of the group-velocity, β_2 is the group-velocity dispersion (GVD), and γ is the Kerr nonlinear coefficient of the fiber. The term of optical loss in the classical NLSE is not included here, because that term is to be treated quantum-mechanically, and derived from the first principles of light-matter interactions. If neglecting the fiber nonlinearity and the GVD, then the NLSE is solved by a space-invariant envelope of the form $A(z,t) = A'(t-\beta_1 z)\exp(i\beta_0 z)$, for some real-valued function $A'(\tau)$. The fiber nonlinearity may be neglected for the current purpose, because the effect of quantum noise is significant only when the signal power becomes low, and that is when the nonlinearity diminishes. The effect of the GVD is not usually negligible, when the signal modulation speed is high and the propagation distance is long. However, it is believed that the GVD would not alter the characteristics of the quantum noise due to the highly localized light-matter interactions, as the accumulation of GVD over a short length of fiber is too small to change the shape of the signal pulse. We shall proceed to quantize the signal field in a single-mode fiber using the space-invariant mode of pulse propagation. Let $q(t) = \cos\omega_0 t$, and $p(t) = \sin\omega_0 t$, then the electric and the magnetic fields are represented as,

$$E(z,t) = E_0[A_r(z,t)q(t) + A_i(z,t)p(t)], \quad (4)$$

$$H(z,t) = H_0[A_i(z,t)q(t) - A_r(z,t)p(t)]. \quad (5)$$

The Hamiltonian, namely the total energy, of the signal field can be calculated as,

$$\mathcal{H} = \frac{1}{2}\int(\epsilon E^2 + \mu H^2)dz = \frac{1}{2}C_1 q^2 + \frac{1}{2}C_2 p^2, \quad (6)$$

with

$$C_1 = \int[\epsilon E_0^2 A_r^2(z,t) + \mu H_0^2 A_i^2(z,t)]dz, \quad (7)$$

$$C_2 = \int[\epsilon E_0^2 A_i^2(z,t) + \mu H_0^2 A_r^2(z,t))]dz. \quad (8)$$

The orthogonality between $A_r(z,t)$ and $A_i(z,t)$ has been used in the derivation. Based on the quadratic form of the Hamiltonian, and the dynamical equations $\dot{q} = -\omega_0 p$, $\dot{p} = \omega_0 q$, we would draw an analogy between the optical mode and a harmonic oscillator, and further the analogy into the quantum world by turning q and p into operators, and introducing commutation relations,

$$[q,p] = \frac{i\hbar\omega_0}{\sqrt{C_1 C_2}}, \quad [q,q] = [p,p] = 0. \quad (9)$$

It is customary to make a canonical transformation to the annihilation and creation operators,

$$\mathsf{a} = q\sqrt{\frac{C_1}{2\hbar\omega_0}} + ip\sqrt{\frac{C_2}{2\hbar\omega_0}}, \quad (10)$$

$$\mathsf{a}^+ = q\sqrt{\frac{C_1}{2\hbar\omega_0}} - ip\sqrt{\frac{C_2}{2\hbar\omega_0}}, \quad (11)$$

which satisfy the commutation relations,

$$[\mathsf{a}, \mathsf{a}^+] = 1, \quad [\mathsf{a}, \mathsf{a}] = [\mathsf{a}^+, \mathsf{a}^+] = 0. \tag{12}$$

The Hamiltonian of the field becomes,

$$\mathcal{H} = \hbar\omega_0 \left(\mathsf{a}^+\mathsf{a} + \frac{1}{2} \right), \tag{13}$$

and the electromagnetic fields, now also operators, are represented as,

$$E(z,t) = E_0 \sqrt{\frac{\hbar\omega_0}{2}} \left[\left(\frac{A_r(z,t)}{\sqrt{C_1}} - i\frac{A_i(z,t)}{\sqrt{C_2}} \right) \mathsf{a} + \left(\frac{A_r(z,t)}{\sqrt{C_1}} + i\frac{A_i(z,t)}{\sqrt{C_2}} \right) \mathsf{a}^+ \right], \tag{14}$$

$$H(z,t) = H_0 \sqrt{\frac{\hbar\omega_0}{2}} \left[\left(\frac{A_r(z,t)}{\sqrt{C_1}} + i\frac{A_i(z,t)}{\sqrt{C_2}} \right) \mathsf{a} + \left(\frac{A_r(z,t)}{\sqrt{C_1}} - i\frac{A_i(z,t)}{\sqrt{C_2}} \right) \mathsf{a}^+ \right], \tag{15}$$

in terms of a and a^+. In the standard quantum theory of light,[3–5] the modal functions are usually time-independent, and often called the normal modes of the field. Here, however, our modal function $A(z,t)$ is time-dependent, albeit slowly. The propagating mode $A(z,t)$ is actually a linear superposition of many normal (time-independent) modes with slightly different oscillation frequencies ω around the center ω_0. By grouping the normal modes together into $A(z,t)$, we have adopted an approximation that neglects the energy variation of $\hbar\omega$ from $\hbar\omega_0$. We shall further assume that all the normal modes participate into the same interactions with the same environment, namely, the same material molecules and unguided modes of the fiber, and the strength of the interactions is approximately the same for all the normal modes. Such approximation enables the concise single-mode quantum description of the optical signal as in equations (12) through (15), and it should be applicable to practical wavelength-division multiplexed (WDM) communication systems, as the modulation bandwidth of the optical signals is usually far less than the center carrier frequency.

The eigen states of a and a^+ are the number states $|n\rangle$, $n \in \mathbf{N} = \{0, 1, 2, \cdots\}$, and the eigen relations are,

$$\mathsf{a}|n\rangle = \sqrt{n}\ |n-1\rangle, \quad \forall\ n \in \mathbf{N}, \tag{16}$$

$$\mathsf{a}^+|n\rangle = \sqrt{n+1}\ |n+1\rangle, \quad \forall\ n \in \mathbf{N}. \tag{17}$$

For each $n \in \mathbf{N}$, the state $|n\rangle$ has the physical interpretation of n photons being contained in the single-mode wave packet defined by $A(z,t)$. The wave function of a general wave packet may be expanded in the basis of the number states as,

$$|\psi\rangle = \sum_n c_n |n\rangle, \tag{18}$$

from which the expectation value of an operator Q may be calculated as,

$$\langle\psi|Q|\psi\rangle = \sum_m \sum_n c_m^* c_n \langle m|Q|n\rangle. \tag{19}$$

In particular, the average photon number is calculated as,

$$\langle\psi|n|\psi\rangle = \langle\psi|\mathsf{a}^+\mathsf{a}|\psi\rangle = \sum_n n|c_n|^2. \tag{20}$$

If the wave packet is probed by a photon counter, for example a photo-detector (which is totally destructive though), $|c_n|^2$ is obviously the probability of n photons being detected.

2. INTERACTION WITH THE ENVIRONMENT

The loss or gain to the wave packet of a single pulse is due to interactions with the environment, including molecules in the waveguide, that may absorb photons from or emit photons to the wave packet, and other optical modes, which signal photons may be scattered into or from. Such interacting molecules and leaky optical modes shall be called interaction centers for convenience. The active molecules and Rayleigh scattering centers[8] are naturally localized in the waveguide material. Light scattering due to waveguide non-uniformities, such as micro-bends and tensile stresses,[8] may take place within an extended length of fiber. However, the effect may still be viewed as localized comparing to the long distance of signal transmission. With all interaction centers localized, the following model Hamiltonian[9] may be used to describe the interaction between the signal field and the environment,

$$U = \sum_k \hbar (g_k \mathsf{a} \sigma_k^+ + g_k^* \mathsf{a}^+ \sigma_k^-) \delta(z - z_k), \tag{21}$$

where $k \in \mathbf{Z}$ labels the interaction centers, z_k is the location of the kth center, g_k is the coupling strength, σ_k^+ and σ_k^- are operators to change the state of the interaction center after absorbing/emitting a photon respectively. It is assumed that the sequence $\{z_k\}_{k\in\mathbf{Z}}$ is a realization of a generally inhomogeneous Poisson point process[10] with intensity $\lambda(z)$ along the length of the fiber, that is, the probability of having an interaction center inside an interval $[z, z + dz)$ is $\lambda(z)dz$ for an infinitesimal dz. Although not always true, it is often a very good approximation to model each interaction center as a two-state system, because multi-photon processes are usually rare events. A general quantum state of the kth interaction center is $d_k|\downarrow\rangle_k + u_k|\uparrow\rangle_k$, where $|d_k|^2 + |u_k|^2 = 1$, $|\downarrow\rangle_k$ and $|\uparrow\rangle_k$ are the down and up states of the interaction center, which satisfy,

$$\begin{aligned} \sigma_k^+ |\downarrow\rangle_k = |\uparrow\rangle_k, \qquad \sigma_k^+ |\uparrow\rangle_k = 0, \\ \sigma_k^- |\uparrow\rangle_k = |\downarrow\rangle_k, \qquad \sigma_k^- |\downarrow\rangle_k = 0. \end{aligned} \tag{22}$$

Despite the simplicity of the signal field and the individual interaction centers, it becomes rather complicated to describe the whole system in a fully quantum-mechanical manner, because of the vast Hilbert space of an interacting many-body system. To shorten the notation, we write the wave function of the whole system in a simplified form,

$$|\Psi(t) = \sum_n \phi_n(t)|n\rangle, \tag{23}$$

where $|n\rangle$ is the nth number state of the signal field, and the coefficient $\phi_n(t)$ is Φ-valued, representing the quantum state of all interaction centers entangled with $|n\rangle$ at time t, Φ is a suitable Hilbert space to accommodate all the possible wave functions of the interaction centers, or simply called the environment. The Schrödinger equation is still of the form,

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = U|\Psi(t)\rangle = \sum_k (g_k \mathsf{a} \sigma_k^+ + g_k^* \mathsf{a}^+ \sigma_k^-) \delta(z - z_k)|\Psi(t)\rangle, \tag{24}$$

in the interaction picture, although the solution is quite involved. Before going further, it is noted that the space-invariance of the wave packet, $A(z, t) = A'(t - \beta_1 z) \exp(i\beta_0 z)$, makes the space and time variables interchangeable, if the space-time extent of the wave packet is neglected, and the signal pulse is represented by a "point particle" at the center of the packet, $t - \beta_1 z = 0$. With t substituted by $\beta_1 z$, equation (24) becomes,

$$i\frac{\partial}{\partial z}|\Psi(z)\rangle = \sum_k \beta_1 (g_k \mathsf{a} \sigma_k^+ + g_k^* \mathsf{a}^+ \sigma_k^-) \delta(z - z_k)|\Psi(z)\rangle, \tag{25}$$

which describes a quantum dynamics along the z axis. Equations (23) and (25) may be combined to get,

$$i\sum_n \dot{\phi}_n(z)|n\rangle = \sum_k \sum_n \beta_1 (g_k \mathsf{a} \sigma_k^+ + g_k^* \mathsf{a}^+ \sigma_k^-) \delta(z - z_k) \phi_n(z)|n\rangle. \tag{26}$$

It is difficult in general to solve the whole system quantum-mechanically. Just the initial condition would be hard to specify, with so many interaction centers, each of which is randomly located, and may be at any superposition state of the two levels. However, we note again that the signal field interacts with the interaction centers in a highly localized manner. The signal can interact with only one center at any given time. The interaction centers that the signal has already passed do not interact with the signal any longer in terms of energy exchange. But the spooky quantum entanglement[11] still connects the signal to the past interaction centers. Were not for the quantum entanglement, a much simpler system of the signal field exchanging energy with the immediate interaction center would be isolated from the complicated many-body system. One way to extract such isolated system from the whole, somewhat forcedly, is to use the density matrix[12] description of the entire system, that is $\rho_{\rm tot} = |\Psi\rangle\langle\Psi|$, then take the trace over all the unwanted degrees of freedom, leaving only those of the signal field and the single chosen interaction center. The reduced density matrix is then a complete representation of the isolated system, although the quantum coherence in the reduced system is also reduced, even totally lost. That is, the system is no longer in a pure quantum state, but a statistically mixed one.[12] This is the familiar yet hard-to-understand phenomenon of decoherence. Fortunately, decoherence is indeed what happens in reality. In fact, the technique of tracing over the environmental degrees of freedom has been used to explain the mysterious phenomenon of decoherence of open quantum systems in general.[13, 14] Furthermore, the main goal of optical communications is not to perform any delicate experiments of quantum optics where quantum coherence needs to be carefully preserved, but to deliver information encoded in the number of photons for which quantum coherence is not the first concern. Consequently, we shall assume that the vast number of interaction centers and their interactions with the larger environment destroy the quantum coherence in a signal wave packet quickly and completely, so that the reduced density matrix of the signal field has only diagonal terms, that is, $\rho = \sum_n \rho_{nn}|n\rangle\langle n|$, where $\rho_{nn} \geq 0$ is a classical probability of the wave packet being at the number state $|n\rangle$. Note that each $|n\rangle$ is still a pure quantum state, but there is no quantum coherence among the number states. So we need only to consider the reduced system of a wave packet at a number state $|n\rangle$ interacting with a single interaction center, as shown in Fig.1, then mix different initial number states statistically according to the input density matrix. An output density matrix is obtained, also in the diagonal form, when taking the trace of the resulted density matrix again over the degrees of freedom of the interaction center. This leads to a classical Markovian model that relates the input/output density matrices.

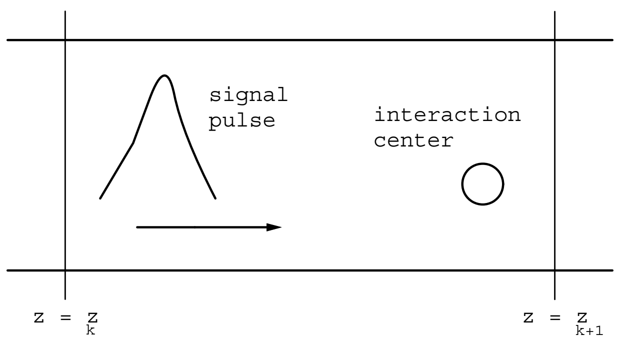

Figure 1. A signal pulse interacting with an interaction center.

3. A CLASSICAL MARKOVIAN MODEL

Let us set $z_{k+1} - z_k = 1/\lambda(z_k)$ in Fig.1, assume that there is one and only interaction center in the interval, which is at a general quantum state $d_k|\downarrow\rangle + u_k|\uparrow\rangle$, and take $|n\rangle$ as the input state of the signal wave packet. The interaction Hamiltonian is $V_k = \hbar(g_k \mathsf{a}\sigma_k^+ + g_k^* \mathsf{a}^+\sigma_k^-)\delta(z - z_k)$, which transforms the input quantum state $\Psi_k = d_k|n\rangle|\downarrow\rangle + u_k|n\rangle|\uparrow\rangle$ into,

$$
\begin{aligned}
\Psi_{k+1} &= \exp[-i\textstyle\int V(t)dt/\hbar]\,(d_k|n\rangle|\downarrow\rangle + u_k|n\rangle|\uparrow\rangle) \\
&= \exp[-i\beta_1(g_k\mathsf{a}\sigma_k^+ + g_k^*\mathsf{a}^+\sigma_k^-)]\,(d_k|n\rangle|\downarrow\rangle + u_k|n\rangle|\uparrow\rangle) \\
&\approx [1 - i\beta_1(g_k\mathsf{a}\sigma_k^+ + g_k^*\mathsf{a}^+\sigma_k^-)]\,(d_k|n\rangle|\downarrow\rangle + u_k|n\rangle|\uparrow\rangle) \\
&= d_k|n\rangle|\downarrow\rangle + u_k|n\rangle|\uparrow\rangle - i\beta_1 g_k d_k\sqrt{n}\;|n-1\rangle|\uparrow\rangle - i\beta_1 g_k^* u_k\sqrt{n+1}\;|n+1\rangle|\downarrow\rangle. \qquad (27)
\end{aligned}
$$

By taking the trace of $|\Psi_{k+1}\rangle\langle\Psi_{k+1}|$ over the interaction-center degrees of freedom, then neglecting the off-diagonal terms, and renormalizing the coefficients, a reduced density matrix for the signal wave packet is obtained as,

$$(1 - n\beta_1^2|g_k|^2 - \beta_1^2|g_k|^2|u_k|^2)|n\rangle\langle n| + n\beta_1^2|g_k|^2|d_k|^2|n-1\rangle\langle n-1| + (n+1)\beta_1^2|g_k|^2|u_k|^2|n+1\rangle\langle n+1|. \quad (28)$$

A mixed state of the signal field $\rho(z_k) = \sum_n \rho_{nn}(z_k)|n\rangle\langle n|$ is therefore converted by the interaction center into another mixed state,

$$\begin{aligned}\rho(z_{k+1}) &= \sum_n (1 - n\beta_1^2|g_k|^2 - \beta_1^2|g_k|^2|u_k|^2)\rho_{nn}(z_k)|n\rangle\langle n| \\ &+ \sum_n (n+1)\beta_1^2|g_k|^2|d_k|^2\rho_{n+1,n+1}(z_k)|n\rangle\langle n| \\ &+ \sum_n n\beta_1^2|g_k|^2|u_k|^2\rho_{n-1,n-1}(z_k)|n\rangle\langle n|. \end{aligned} \quad (29)$$

By definition, $\rho(z_{k+1}) = \sum_n \rho_{nn}(z_{k+1})|n\rangle\langle n|$. So the coefficients of the density matrices are related by,

$$\begin{aligned}\rho_{nn}(z_{k+1}) &= (1 - n\beta_1^2|g_k|^2 - \beta_1^2|g_k|^2|u_k|^2)\rho_{nn}(z_k) + (n+1)\beta_1^2|g_k|^2|d_k|^2\rho_{n+1,n+1}(z_k) \\ &+ n\beta_1^2|g_k|^2|u_k|^2\rho_{n-1,n-1}(z_k), \quad \forall\, n \in \mathbf{N}. \end{aligned} \quad (30)$$

The dynamics of the reduced density matrix may be interpreted as a discrete Markov chain[15–17] indexed by $k \in \mathbf{Z}$, with $\mathbf{N} = 0, 1, 2, \cdots$ being the state space, and $[\rho_{00}(z_k), \rho_{11}(z_k), \rho_{22}(z_k), \cdots]$ being the probability distribution vector at "time" k. The compound process of the discrete Markov chain and the Poisson point process of interaction centers is a continuous Markov chain along the z axis, with the transition law for the probability distribution given by a continuous version of equation (30),

$$\dot\rho_{nn}(z) = -\alpha(z)[n + f(z)]\rho_{nn}(z) + \alpha(z)[1 - f(z)](n+1)\rho_{n+1,n+1}(z) + \alpha(z)f(z)n\rho_{n-1,n-1}(z), \quad \forall\, n \in \mathbf{N}, \quad (31)$$

where

$$\alpha(z) \stackrel{\text{def}}{=} \lambda(z)\beta_1^2|g(z)|^2, \quad (32)$$

$$f(z) \stackrel{\text{def}}{=} |u(z)|^2. \quad (33)$$

Obviously, $\alpha(z) \geq 0$ represents a compound interaction strength, which is the spatial density of interaction centers times their quantum coupling strength to the signal field, $0 \leq f(z) \leq 1$ is the fraction of interaction centers at z that are at the excited state, ready to emit a photon. Note that our definition of the interaction centers includes both the actual atomic levels of the active molecules, and the passive light scattering into/from other optical modes due to fiber non-uniformities, micro-bends, and tensile stresses *etc.* Consequently, the parameters $\alpha(z)$ and $f(z)$ should be determined by counting both the active and the passive interaction centers. The effect of the passive interaction centers may become significant in, for example, distributed Raman and erbium-doped fiber amplifiers, where the passive centers could induce a sizable internal loss in the fiber. The commonplace case of a transmission fiber with no amplification in the middle constitutes an extreme in which the passive light scatterers all at the "down state" are the only interaction centers.

So we have derived, from the first-principles of quantum optics, the fundamental equation (31) governing the dynamics of photon-number distribution of optical signals propagating in a waveguide with loss and/or gain. The mathematical equation is essentially the same as appeared in precious studies of photon statistics in optical amplifiers.[18–20] It may be recognized as the forward Kolmogorov equation,[16, 17] and can be solved analytically by using the method of probability generating function (PGF).[18–20] To simplify the notation, let

$$P_n(z) = \rho_{nn}(z), \quad \forall\, n \in \mathbf{N}, \quad (34)$$

$$a(z) = \alpha(z)f(z), \quad (35)$$

$$b(z) = \alpha(z)[1 - f(z)], \quad (36)$$

then equation (31) may be re-written as,

$$\dot P_n = a[nP_{n-1} - (n+1)P_n] + b[(n+1)P_{n+1} - nP_n], \quad (37)$$

which is in the same form as appeared in previous works.[20] The established formulas[18–20] can then be adopted with little or no modification. A brief introduction to the PGF method seems to be appropriate here. A PGF $F(x,z)$ for the probability distribution vector $[P_0(z), P_1(z), P_2(z), \cdots]$ is defined as,

$$F(x,z) \stackrel{\text{def}}{=} \sum_n x^n P_n(z). \tag{38}$$

And the inverting formula is,

$$P_n(z) = \frac{1}{n!}\left[\frac{\partial^n F(x,z)}{\partial x^n}\right]_{x=0}, \quad \forall\, n \in \mathbf{N}. \tag{39}$$

The forward Kolmogorov equation (37) may be translated into a differential equation for the PGF,

$$\frac{\partial F}{\partial z} = (ax-b)(x-1)\frac{\partial F}{\partial x} + a(x-1)F, \tag{40}$$

with the initial condition,

$$F(x,0) = \sum_n x^n P_n(0). \tag{41}$$

The differential equation is analytically solved by,[18–20]

$$F(x,z) = \frac{1}{1+N(1-x)}\sum_n P_n(0)\left[\frac{1+(N-G)(1-x)}{1+N(1-x)}\right]^n, \tag{42}$$

where

$$G(z) \quad \stackrel{\text{def}}{=} \quad \exp\left[\int_0^z [a(\zeta)-b(\zeta)]\, d\zeta\right], \tag{43}$$

$$N(z) \quad \stackrel{\text{def}}{=} \quad G(z)\int_0^z \frac{a(\zeta)}{G(\zeta)} d\zeta. \tag{44}$$

The physical interpretation of $G(z)$ is the overall gain/loss from 0 to z, whereas $N(z)$ may be interpreted as the ASE due to the presence of interaction centers that are at the "up state".

4. APPLICATIONS AND A NUMERICAL EXAMPLE

Our Markovian model is derived rigorously from the first principles of quantum optics, and the model is applicable to a wide range of guided-wave systems with arbitrary distributions of gain/loss media along the length of the waveguide. The existence of analytical solutions enhances further the appeal and prediction power of the established model. We expect the model to find many applications in quantifying the quantum noise in fiber-optic systems, such as transmission fibers without gain, doped fiber amplifiers, and Raman amplifiers. As an example, we shall work out the quantum noise induced by the pure loss of a transmission fiber. This example is chosen not just because of its simplicity, but more importantly due to the fact that such noise has largely been neglected by the fiber-optic engineering community.

Optical signals usually start with a high initial power level and high signal-to-noise ratio (SNR) in order to reach a long transmission distance. For all practical purposes, the noise of the laser transmitter may be neglected, and the starting signal may be modelled by a pure number state $|m\rangle$, which is of course noise-free if detected by an ideal photo-detector. The photon number m corresponds to the total energy of a pulse. For example, in a 10 Gb/s system, the peak optical power is often set to approximately 0 dBm, or 1 mW, so a binary "one" is represented by an energy packet of 1 mW $\times$ 100 ps $= 10^{-13}$ J, or approximately 10^6 photons at 1.55 μm. So the input state of the pulse may be set to the number stat $|m\rangle$ with $m = 10^6$. The average photon number will be reduced to about 10^4 after a 100-km fiber propagation with 20 dB loss. Because of the random nature of the loss process, the actual photon number at the end will fluctuate around 10^4, even though the signal starts with exactly 10^6 photons. We resort to equation (42) for an exact solution to the probability distribution of the number of signal photons at the output of the transmission fiber. There is only loss in this case, $G = 0.01$ and $N = 0$, therefore,

$$F(x) = \sum_n P_n(0)\,[(1-G)+Gx]^n = [(1-G)+Gx]^m\,, \tag{45}$$

which corresponds to the well-known binomial distribution,

$$P_n = \begin{pmatrix} m \\ n \end{pmatrix} (1-G)^{m-n}G^n, \quad \forall\; n \in [0,m], \tag{46}$$

here $m = 1,000,000$ is a very large number. This result agrees exactly with that obtained using a Langevin noise operator in the Heisenberg representation.[1] The probability distribution is plotted in Fig.2, where the horizontal axis is the deviation of the photon number n from the mean $mG = 10,000$, and the vertical axis is the normalized probability P_n. Graphically, the signal photon number is seen to fluctuate on the order of ± 100 with large probability. To quantify the effect of the quantum noise, it is found that the probability distribution is excellently fitted by a Gaussian distribution,

$$P_n \approx P(n) = P_{\max}\exp\left[-\frac{(n-mG)^2}{2mG(1-G)}\right], \tag{47}$$

here $G = 0.01$ and $mG = 10,000$. Indeed, it can be proved by using Stirling's formula that $\log P(n)$ is the Taylor expansion of $\log P_n$ in power series of $(n-mG)/\sqrt{mG(1-G)}$ up to the quadratic term.[21] When the attenuated optical signals are converted into electrical current by an ideal photo-detector, the level of the "one" bits is Gaussian distributed as given above, while that of the "zero" bits is free of fluctuations, so the Q factor is,[1]

$$Q \stackrel{\text{def}}{=} \frac{\langle n\rangle_1 - \langle n\rangle_0}{\sigma_1+\sigma_0} = \frac{\langle n\rangle_1}{\sigma_1} = \frac{mG}{\sqrt{mG(1-G)}} = \sqrt{\frac{mG}{1-G}}. \tag{48}$$

For the current example, it predicts $Q \approx 100$ after one span of 100-km fiber. However, the Q factor will decrease quickly as the transmission distance increases. Even if the attenuated signal were boosted back to the 1-mW power level by a noiseless amplifier, and such amplified fiber span were repeated M times to reach a long-distance of $M \times 100$ km, the Q would be degraded to $\sqrt{mG/M(1-G)}$, based on a model assuming that the "effective Gaussian noise" at the end of each fiber span is independent and additive. The independency assumption is only natural, while the additivity should be a good approximation so long as the accumulated noise remains much lower than the signal level. If taking $M = 25$ for a 2500-km transmission line, the Q would not be much better than 20 at the end, even completely neglecting the noise contribution of the repeating amplifiers! The problem gets worse when the signal modulation speed goes to 40 Gb/s and higher. If the signal power level is fixed, then the number of photons contained in one pulse is inversely proportional to the modulation speed. Consequently, the Q is inversely proportional to the square-root of the modulation speed, so $Q \leq 10$ for 40 Gb/s, and $Q \leq 5$ for 160 Gb/s, at the end of 25 fiber spans of 20 dB loss. The Q factor needs to be higher than 6 in order to guarantee a bit-error-rate (BER) below 10^{-9}.[1] Clearly, the quantum noise due to fiber loss could amount to a significant source of noise that should be seriously considered in practical fiber transmission systems, especially in those with modulation speed of 40 Gb/s and above.

REFERENCES

1. H. A Haus, *Electromagnetic Noise and Quantum Optical Measurements*. Springer-Verlag, Berlin, 2000.
2. J. R. Jeffers, N. Imoto, and R. Loudon, "Quantum optics of traveling-wave attenuators and amplifiers," *Phys. Rev. A*, vol. 47, no. 4, pp. 3346 - 3359, 1993.
3. R. Loudon, *The Quantum Theory of Light*, 3rd ed. Oxford University Press, 2000.
4. W. Heitler, *The Quantum Theory of Radiation*, 3rd ed. Clarendon Press, Oxford, 1954.
5. P. A. M. Dirac, "The quantum theory of the emission and absorption of radiation," *Proc. Roy. Soc. Lond. A*, vol. 114, pp. 243 - 265, 1927.
6. G. P. Agrawal, *Nonlinear Fiber Optics*, 2nd ed. Academic Press, San Diego, 1995.

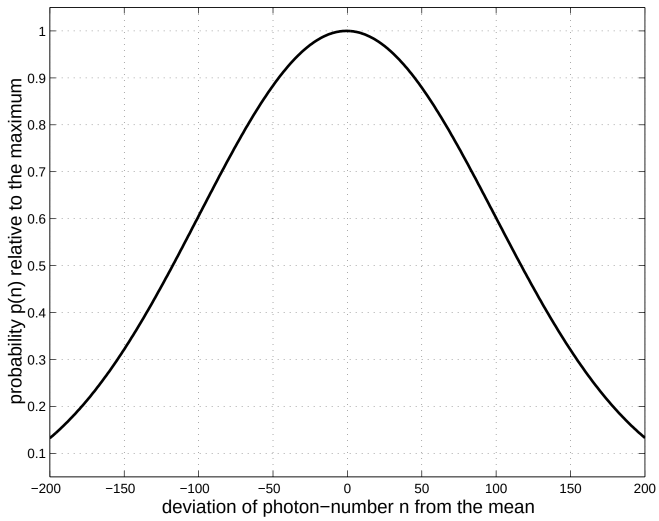

Figure 2. Probability distribution of the number of photons after 100-km fiber transmission.

7. H. Wei and D. V. Plant, "Fundamental equations of nonlinear fiber optics," to be presented at *SPIE's 48th Annual Meeting*, San Diego, CA, August 3 - 8, 2003.
8. A. W. Snyder and J. D. Love, *Optical Waveguide Theory*. Chapman and Hall, Now York, 1983.
9. M. O. Scully and M. S. Zubairy, *Quantum Optics*. Cambridge University Press, Cambridge, 1997.
10. J. F. C. Kingman, *Poisson Processes*. Clarendon Press, Oxford, 1993.
11. A. D. Aczel, *Entanglement: The Greatest Mystery in Physics*. Four Walls Eight Windows, New York, 2001.
12. K. Blum, *Density Matrix Theory and Applications*, 2nd ed. Plenum Press, New York, 1996.
13. W. H. Zurek, "Decoherence and the transition from quantum to classical," *Physics Today*, vol. 44, pp. 36 - 44, 1991.
14. R. Omnes, *The Interpretation of Quantum Mechanics*. Princeton University Press, Princeton, NJ, 1994.
15. D. Freedman, *Markov Chains*. Springer-Verlag, New York, 1983.
16. S. Karlin and H. M. Taylor, *A First Course in Stochastic Processes*, 2d ed. Academic Press, San Diego, 1975.
17. S. Karlin and H. M. Taylor, *A Second Course in Stochastic Processes*. Academic Press, New York, 1981.
18. K. Shimoda, H. Takahasi, C. H. Townes, "Fluctuations in amplification of quanta with application to maser amplifiers," *J. Phys. Soc. Japan*, vol. 12, no, 6, p. 686, 1957.
19. P. Diament and M. C. Teich, "Evolution of the statistical properties of photons passed through a traveling-wave laser amplifier," *IEEE J. Quan. Elec.*, vol. 28, no. 5, p. 132, 1992.
20. E. Desurvire, *Erbium-Doped Fiber Amplifiers: Principles and Applications*. John Wiley & Sons, New York, 1994.
21. http://mathworld.wolfram.com/BinomialDistribution.html